

Exercise 3

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$1 + xe^x - e^x = \int_0^x tu(t) dt$$

Solution

We seek a series solution for u :

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

Substitute this and the Taylor series expansion of e^x ,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots,$$

into the integral equation.

$$\begin{aligned} 1 + x \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots \right) \\ - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \dots \right) \\ = \int_0^x t(a_0 + a_1t + a_2t^2 + a_3t^3 + \dots) dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\begin{aligned} 1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{120} + \dots \\ = \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4 + \frac{a_3}{5}x^5 + \dots \\ \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \dots = \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_2}{4}x^4 + \frac{a_3}{5}x^5 + \dots \end{aligned}$$

Match the coefficients of the respective powers of x to determine a_i .

$$\begin{aligned} \frac{a_0}{2} = \frac{1}{2} & \rightarrow a_0 = 1 \\ \frac{a_1}{3} = \frac{1}{3} & \rightarrow a_1 = 1 \\ \frac{a_2}{4} = \frac{1}{8} & \rightarrow a_2 = \frac{1}{2} \\ \frac{a_3}{5} = \frac{1}{30} & \rightarrow a_3 = \frac{1}{6} \\ \vdots & \quad \quad \quad \vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ &= e^x. \end{aligned}$$